A STUDY OF COMMUTATIVITY AND STRUCTURES OF CERTAIN RINGS

THESIS

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CERTIFICATE

This is to certify that the thesis entitled ***" A STUDY OF COMMUTATIVITY AND STRUCTURES OF CERTAIN RINGS"*** *is based on a part of the research work of* ***Ms. Achlesh Kumari*** *carried out under my guidance in theDepartment of Mathematics Aligarh Muslim University, Aligarh. To the best of my knowledge, the work included in the thesis is original and has not* been *submitted to any other University or Institution for the award of a degree.*

It is further certified that **Ms. Achlesh Kumari** *has fulfilled the prescribed conditions of duration and nature given in the statutes and ordinances of the Aligarh Muslim University, Aligarh.*

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Preface

Although the celebrated theorem due to Weddereburn was established as early as 1905, it was during the second half of the last century after development of the structure theory of rings that striking results relating to commutativity of rings, started appearing in mathematical literature. The theorem to which we refer is namely: " *any finite division ring is necessarily commutative*". Besides various generalizations of this theorem, research works of Jacobson, McCoy, Faith, Kaplansky and Herstein etc. in this direction have attracted a large number of algebraists to investigate the commutativity of rings under certain given conditions. The present thesis entitled "**A Study of Commutativity and Structures of Certain Rings**" includes the same type of work carried out by the author during the past few years at the Department of Mathematics Aligarh Muslim University, Aligarh.

The thesis comprises five chapters, which are further divided into different articles(sections). In Chapter 1, we have collected some basic definitions, ring theoretic concepts and well-known results relevant to our study which may be needed for the development of the subsequent text. No attempt is, however, made to provide the proofs of these known results or to include the elementary algebraic concepts such as rings, ideals, homomorphisms, fields and modules etcetera.

In Chapter 2, we extend some commutativity theorems due to I.N. Herstein. In an attempt to generalize the above mentioned theorem of Weddereburn and also the result that any Boolean ring is commutative, Jacobson proved in 1945 that *any ring R in which* $x^{n(x)} = x$, $n(x)$ *an* *integer greater than* 1, *for every element* x *in the ring, must be commutative.* This result has been generalized in different directions. Among others, Kaplansky in 1951 and Faith in 1960 weakened Jacobson's condition by assuming that $x^{n(x)} \in Z(R)$, the center of R, for all $x \in R$ and established commutativity of semi-prime rings and division rings respectively. In 1953, Herstein further sharpened Jacobson's result in other directions of research (cf. theorem H_1 and theorem H_2). In § 2.2, we extend these results for a ring with unity by imposing torsion condition on the ring. In the next section, the study has been further extended. Another result of Herstein asserts that *if the map* $x \longrightarrow x^n$ *with fixed integer* n *defines an endomorphism for the additive abelian group of a ring* R, *then the commutator ideal of* R *must be nil and set of all nilpotent elements of* R *is an ideal.* In § 2.4, we observe that the ring satisfying above condition also enjoy the property (B) : $[x^n, y] = [x, y^n]$, for all $x, y \in R$ and establish commutativity of rings with unity satisfying a much wider condition than (B). Infact, we examine the commutativity under the condition (P) : let $n > 1$ and q be fixed non negative integers and R be a ring in which for each $y \in R$, there exist integers $m = m(y) \geq 0$ and $r = r(y) \geq 0$ such that $x^q[x^n, y] = [x, y^m]y^r$, for all $x \in R$.

There are sufficient examples (cf. example 2.3.4) to demonstrate that certain conditions which render a ring with unity commutative, fail to yield commutativity for rings without unity. The algebraists like Tominaga, Hirano and Kamatsu etcetera initiated the study of s-unital and one sided s-unital rings which constitute comparatively larger classes of rings than that of rings with unity and extend some commutativity results valid for rings with unity to these wider classes of rings. Chapter 3 of our thesis deals with some commutativity conditions for s-unital and one sided s-unital rings. In § 3.2, a few preliminaries and basic results related to these rings are collected. In § 3.3, the result of theorem 2.4.3 has been extended to one sided s-unital rings while in § 3.2 , we generalize a result of Ashraf, Quadri and Ali [18]. The same result has been further generalized in the last section using a classification for non-commutative rings due to W. Streb [116] as a tool.

A concept that generalizes Boolean rings (satisfying $x^2 = x$) as well as J-rings (satisfying $x^{n(x)} = x$) is that of Periodic rings (satisfying $x^{n(x)} = x^{m(x)}$, for distinct positive integers $n(x)$ and $m(x)$). These rings have been among the favourite of many ring theorists over the last few decades ————— ; to mention a few: Chacron [43], Bell [29] & [33], Searcoid and MacHale [114], Ligh and Luh [92], Bell and Ligh [34], Grosen et al. [48] etc. In Chapter 4, we study direct sum decompositions of rings under certain polynomial constraints which, infact lead to buy periodicity of the rings. § 4.2, is devoted to collect some preliminaries related to periodic rings. In § 4.3 , we obtain structures of rings satisfying the properties (i) $xy = f(y)x^{n(x,y)}$ (ii) $xy = f(x)y^{n(x,y)}$ (iii) $xy = x^{n(x,y)}f(y)$ (iv) $xy = y^{n(x,y)}f(x)$ where $f(\lambda) \in \lambda^2\mathbb{Z}[\lambda]$. The commutativity of such rings is deduced as corollary of the structure theorem. The result of § 4.3 is further generalized in the next section by establishing the decomposition of the ring when the underlying conditions are assumed to be satisfied by certain appropriate subset of R. In the last section of the chapter, suitable examples are provided to justify the restrictions imposed on the hypotheses of the results proved in previous sections.

Recently many algebraists have studied commutativity of prime and semi-prime rings admitting certain special types of functions such as commutativity preserving (cp) and strong commutativity preserving (scp) (cf. definitions 5.2.3 and 5.3.4) derivations and endomorphisms. Inspired by these works, we initiate in Chapter 5, the study of co-commutativity preserving (*ccp*) and srtong co-commutativity preserving (*sccp*)-mappings, the concepts more general than (*cp*) and (*scp*)-mappings. As in the previous chapters, we recall some important notions and known results related to the mentioned study in § 5.2. In the next section, we prove that *if a semi-prime ring R with a nonzero ideal A of R admits two (sccp)-derivations, then R contains a nonzero central ideal.* Also, in the process we prove that *if, in particular, R is assumed to be prime in the above result, then R is necessarily commutative.* The chapter has been concluded by obtaining a result on (*sccp*)-endomorphisms.

At the end, an exhaustive bibliography of the related literature has been included.

To specify definitions, examples and results throughout the thesis system of double decimal numbering has been adopted. The first figure represents the chapter, the second denotes the article (section) and the third demonstrates the number of the definition, the example, the lemma, the theorem or the corollary as the case may be in a particular chapter. For example, theorem 4.3.2 refers to second theorem of chapter 4 appearing in section 4.3.

Some papers based on the portions of the text have either already been published or accepted for publication in standard refereed Mathematical Journals/Research volumes.

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CHAPTER - 1

PRELIMINARIES

§1.1 INTRODUCTION.

The present chapter includes some basic notions and important terminology which we shall need for the development of the subject in the subsequent chapters of our thesis. However, the elementry knowledge of the algebric concepts like groups, rings, ideals, fields and homomorphisms etcetera has been preassumed. Some key results and classical theorems related to our subject matter are also incorporated for ready reference. At places, suitable examples and necessary remarks are given to make the exposition as self contained as possible.

The most of the material presented in this chapter has been extracted from the standard texts like Behrens and Ries [22], Herstein ([57], [58]), Jacobson [68], Kurosh [87], Lambek [88], McCoy [95] and Rowen [113].

§1.2 SOME RING THEORETIC CONCEPTS.

This section is aimed to collect some important terminology in ring theory. Throughtout, R represents an associative ring (may be without unity) until otherwise mentioned. For any pair of elements a, b in R, the commutator $ab - ba$ will be denoted by $[a, b]$ and anti commutator $ab + ba$ by aob.

Definition 1.2.1 (Characteristic of a Ring). The smallest positive integer n (if there is one) such that $na = 0$ for all $a \in R$ is called the characteristic of the ring R. If there exists no such integer, we say that R has characteristic zero. If R has unity 1 then the characteristic n of R is also the smallest positive integer for which $n \cdot 1 = 0$. We shall generally denote the characteristic of R by char R or simply ch R.

Definition 1.2.2 (Idempotent Element). An element e of a ring R is said to be idempotent if $e^2 = e$.

An idempotent is said to be a central idempotent if it commutes with every ring element.

Remark 1.2.1. (i) Trivially zero of a ring R is idempotent. If R contains unity 1, then 1 is also idempotent and $1 - e$ is idempotent whenever e is idempotent.

(ii) If e is a central idempotent, then so is $1 - e$.

Definition 1.2.3 (Nilpotent Element). An element x of R is said to be nilpotent if there exists a positive integer n such that $x^n = 0$. If the element $x \in R$ is nilpotent, then the least positive integer n with $x^n = 0$ is termed as the index of nilpotency of x.

Remark 1.2.2 Every nilpotent element is necessarily a divisor of zero. Indeed if $x \neq 0$ is nilpotent, then there exists the smallest positive integer $n > 1$ such that $x^n = 0$ so that $x(x)^{n-1} = 0$ with $x^{n-1} \neq 0$.

Definition 1.2.4 (Center of a Ring). The center $Z(R)$ of a ring R is the set of all those elements of R which commute with each element of R, i.e. $Z(R) = \{x \in R/xr = rx, \text{ for all } r \in R\}$.

Definition 1.2.5 (Nilpotent Ideal). An ideal A of a ring R is said to be nilpotent if there exits a positive integer n, such that $A^n = (0)$.

Definition 1.2.6 (Nil Ideal). An ideal A of a ring R is said to be nil if every element of A is nilpotent.

Remarks 1.2.3. (i) Nilpotent (nil) left or right ideals and nil rings are also defined in the same fashion above.

(ii) A nilpotent ideal is a nil ideal but converse need not be true. For an example of a nil ideal which is not nilpotent we refer to McCoy [84].

(iii) In a commutative ring, the set of all nilpotent elements is an ideal but this need not be true in non-commutative rings.

Example 1.2.1. Let $\mathbb{Z}_2$ be the ring of all 2×2 matrices over the ring $\mathbb{Z}$ of integers, then $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are nilpotent elements of $\mathbb{Z}_2$ but their sum is not nilpotent. This shows that the set of all nilpotent elements in $\mathbb{Z}_2$ is not closed under addition and as such can not be an ideal (left or right) in $\mathbb{Z}_2$.

Definition 1.2.7 (Commutator Ideal). The commutator ideal $C(R)$ of a ring R is the ideal generated by all Commutators $[x, y]$ with $x, y \in R$.

Definition 1.2.8 (Prime Ideal). An ideal P in a ring R is said to be a prime ideal if and only if it has the property that for any ideals A, B in R whenever $AB \subseteq P$ then $A \subseteq P$ or $B \subseteq P$.

Remarks 1.2.4. (i) An ideal P of R is a prime if and only if for any $a, b \in R$ whenever $aRb \subseteq P$, then $a \in P$ or $b \in P$.

(ii) If R is a commutative ring, then an ideal P of R is a prime ideal if and only if for any $a, b \in R$, whenever $ab \in P$, then $a \in P$ or $b \in P$.

Definition 1.2.9 (Semi-prime Ideal). An ideal Q in a ring R is said to be semi prime if for any ideal A in R, whenever $A^2 \subseteq Q$ then $A \subseteq Q$.

Remarks 1.2.5. (i) A prime ideal is necceserily semi prime also but converse need not hold.

(ii) An ideal Q in R is semi prime iff $aRa \subseteq Q$ implies that $a \in Q$.

(iii) Intersection of prime (semi prime) ideals is semi prime. Thus in the ring $\mathbb{Z}$ of integers ideal $(2) \cap (3) = (6)$ is semi prime which is not prime.

Definition 1.2.10 (Maximal Ideal). An ideal M of a ring R is called maximal if $M \neq R$ and there exists no ideal A in R such that $M \subset A \subset R$.

Remarks 1.2.6. (i) If $M(\neq R)$ is a maximal ideal of R then for any ideal A of $R, M \subseteq A \subseteq R$ holds only when either $A = M$ or $A = R$.

(ii) Every maximal ideasl in a commutative ring with unity 1 is prime. However, the converse of this statement is not valid.

The following example shows that unity in the ring is essential for the validity of the above statement.

Example 1.2.3. The ideal (4) in E, the ring of even integers is maximal, but certainly not prime. Indeed,

$$2.2 \in (4) \text{ but } 2 \notin (4)$$

Definition 1.2.11 (Jacobson Radical). The Jacobson radical $J(R)$ of a ring R is the intersection of all maximal left (right) ideals of R.

Remarks 1.2.7 (i) $J(R)$ is a two sided ideal of R.

(ii) $J(R)$ is the set of all those elements of R which annihilate all the irreducible R-modules i.e.

$$J(R) = \{r \in R \mid rM = 0 \text{ for every irreducible } R-\text{module } M\}.$$

Definition 1.2.12 (Annihilator). If M is a subset of a commutative ring R, then the *annihilator* of M, denoted by $Ann(M)$ is the set of all elements r of R such that $rm = 0$, for all $m \in M$. Thus

$$Ann(M) = \{r \in R \mid rm = 0, \quad \text{for all} \quad m \in M\}.$$

Definition 1.2.13 (Prime Ring). A ring R is said to be prime if its zero ideal (0) is prime. Thus R is prime if and only if for any $a, b \in R$ such that $aRb = (0)$, we have $a = 0$ or $b = 0$.

Equivalently, a ring R is a prime ring if and only if any one of the following holds :

(i) If (a) and (b) are principal ideals in R such that $(a)(b) = 0$, then $a = 0$ or $b = 0$.

(ii) The left annihilator of a nonzero left ideal is (0).

Definition 1.2.14 (Semi Prime Ring). A ring R is said to be a semi prime ring if it has no non-zero nilpotent ideals.

Remark 1.2.8 R is a semi-prime ring if and only if for any $x \in R$, whenever $xRx = (0)$, then $x = 0$.

Definition 1.2.15 (Simple Ring). A ring R is called simple if $R^2 \neq (0)$ and it has no ideals other then (0) and (R) itself.

Definition 1.2.16 (Semi Simple Ring). A ring R with zero Jacobson radical is said to be a semi simple ring.

Definition 1.2.17 (Boolean Ring). A ring R is said to be a Boolean ring if all its elements are idempotent i.e. $x^2 = x$, for all $x \in R$.

Remarks 1.2.8 (i) Every Boolean ring has characteristics 2.
(ii) Every Boolean ring is necessarily commutative.

Definition 1.2.18 (Polynomial Identity). A polynomial $f(X_1, X_2, \ldots, X_t)$ in non-commuting indeterminates $X_1, X_2, \ldots, X_t$ with integral coefficients is said to be a polynomial identity in R if $f(r_1, r_2, \ldots, r_t) = 0$, for every $r_1, r_2, \ldots, r_t \in R$. If $f(x_1, x_2, \cdots, x_t)$ is a polynomial identity, then we may simply say that f is a polynomial identity in R. Some time we also say that R satisfies f.

Definition 1.2.19 (Direct Sum and Subdirect Sum of Rings). Let $S_i, i \in U$ be a family of rings indexed by the set U and let us denote by

S the set of all functions defined on the set U such that for each $i \in U$, the value of the function at i is an element of S_i. If addition and multiplication in S are defined as : $(a+b)(i) = a(i)+b(i), (ab)i = a(i)b(i)$ for $a, b \in S$, then S is ring which is called the complete direct sum of the rings $S_i, i \in U$. The set of all functions in S which take on the values zero at all but at most a finite number of elements i of U is a subring of S which is called discrete direct sum of the rings $S_i, i \in U$. However, if U is a finite set, the complete (discrete) direct sum of rings $S_i, i \in U$, as defined above is called direct sum of the rings $S_i, i \in U$.

Let T be a subring of the direct sum S of S_i and for each $i \in U$ let θ_i be a homomorphism of S onto S_i defined by $a\theta_i = a(i)$ for $a \in S$. If $T\theta_i = S_i$ for every $i \in U, thenT$ is said to be a subdirect sum of the rings $S_i, i \in U$.

§1.3. SOME KEY RESULTS.

In this section we state some well known results which may be frequently refered to in subsequent text. For their proofs, the references are mentioned against respective results for those who develop interests in them.

Theorem 1.3.1 (Wedderburn [122]). A finite division ring is a field.

Theorem 1.3.2 (Jcobson [67]). If in a ring R for every $x \in R$, there exists a positive integer $n(x) > 1$ depending on the element $x \in R$ such that $x^{n(x)} = x.$, then R must be commutative.

Theorem 1.3.3 (Kaplansky [73]). Let R be a ring with center $Z(R)$. Suppose there exists a positive integer $n(x) > 1$ such that $x^{n(x)} \in Z(R)$, for every $x \in R$. If in addition R is semi simple, then it is necessarily commutative.

Theorem 1.3.4 (Faith [46]). Let R be a division ring and $A \neq D$ be a subring of R. If every $x \in R$, there exists a positive integer $n(x) \geq 1$, depending on x with $x^{n(x)} \in A$, then R is a field.

Theorem 1.3.5 (Herstein [51]). If R is a ring with center $Z(R)$ such that $x^{n(x)} - x \in Z(R)$ for all $x \in R$, where $n(x) > 1$ is a positive integer depending on x, then R is commutative.

Theorem 1.3.6 (Herstein [53]). Suppose that R is a ring such that given any two elements $x, y \in R$ there exists some positive integer $n(x, y) \geq 1$ which depends on both x and y satissying

$$x^{n(x,y)}y = yx^{n(x,y)}.$$

Then either R is commutative or its commuator ideal is nil.

Theorem 1.3.7 (Herstein [53]). If for every x and y in a ring R we can find a polynomial $p_{x,y}(t)$ with integers coefficients which depend on x and y such that $x^2 p_{x,y}(x) - x$ commutes with y, then R is commutative.

Theorem 1.3.8 (Herstein [55]). If R is a ring in which the mapping $x \mapsto x^n$ for a fixed integer $n > 1$ is an epimorphisim, then R is commutative.

Theorem 1.3.9 (Herstein [60]). Let R be a ring in which given $a, b \in R$, there exist integers $m = m(a,b) \geq 1$, $n = n(a,b) \geq 1$ such that $a^m b^n = b^n a^m$. Then the commutator ideal of R is nil.

Theorem 1.3.10 (Bell [27]). Let R be a ring satisfying $q(x) = 0$, where $q(x)$ is a polynomial in a finite number of non-commutative indeterminates, its coefficients being integers with highest common factor 1. If there exists no prime p for which the ring of 2×2 matrices over $GF(p)$ satisfies $q(x) = 0$, then R has nil commutator ideal and the nilpotent elements of R form an ideal.

CHAPTER - 2

ON SOME COMMUTATIVITY THEOREMS OF HERSTEIN

§2.1 INTRODUCTION

In this chapter we shall generalize some commutativity results for associative rings due to I. N. Herstein. Nearly twenty five years ago, Herstein [59] reproved a well known theorem of Faith [46] which in turn extends a result due to Kaplansky [73]. The theorem to which we refer is namely : if D is a division ring and $A \neq D$ is a subring of D such that for any $x \in D$, there exists a positive integer $n = n(x) \geq 1$ satisfying $x^n \in A$, then D is neccessarily commutative. In section 2.2 we extend this result, ofcourse for global fixed positive integer n. Infact we prove the following : "Let R be a ring with unity 1 in which there exists a positive integer $n \geq 1$ such that commutators are $!n$ torsion free and x^n is central for every $x \in R$, Then R must be commutative". This result has been further extended in the next section.

In the last section we prove that if in a ring with unity 1, there exists integers $n > 1, p \geq 0$ and $m = m(y) \geq 0$ and $r = r(y) \geq 0$ for any $y \in R$ such that $x^p[x^n, y] = [x, y^m]y^r$ for all $x \in R$, then R is commutative. A close look to this result will reveal that it is related to another commutativity study of Herstein [55]. Suitable examples are provided to justify the restrictions imposed on the hypotheses of the theorems proved in the chapter.

§2.2.

Built into the definition of the center $Z(R)$ of a ring R is the condition that R is commutative if and only if $x \in Z(R)$, for all $x \in R$. It must be therefore natural to ask as to what can be said about the commutativity of the ring if some power of each ring element is central. One should, however not jump to hasty generalizations. This will be revealed by the folowing example.

Example 2.2.1. Let $R = \{(a_{ij}) \mid a_{ij}$ are integers with $a_{ij} = 0$ for $i \leq j\}$ be the set of all 3×3 strictly lower triangular matrices over the ring $\mathbb{Z}$ of integers. Then R is a non-commutative ring in which x^3 is central for each $x \in R$.

In 1951, Kaplansky [73] proved that a semisimple ring R must be commutative if R satisfies the following condition :

$(*)$: There exists a positive integer $n(x) \geq 1$ such that $x^n \in Z(R)$, for every $x \in R$.

Later, Carl Faith [46] obtained an extension of the mentioned result but it was also concerned rather to a more restricted class of rings. Infact, Faith established the result for division rings. One may observed that the ring given in example 2.2.1 is a nil ring of index of nilpotency 3. There could be possibility that the rings in which elements are reasonably well behaved may be commutative under the condition $(*)$. Pherhaps with this observation, Herstein [50, Theorem 4 and 5] proved the following :

Theorem H_1. Let R be a ring such that for each $x \in R$, there exists a positive integer $n(x), \geq 1$ depending on x satisfying $x^{n(x)} \in z(R)$. Then R is not commutative only if every element in the commutator ideal $C(R)$ of R is nilpotent.

Theorem H_1. Let R be as above. If R possesses no non-zero nil ideal, then R must be commutative.

There is another observation about the ring given in the mentioned example that it does not contain unity, One might therefore hope that a ring with unity may be commutative under the condition $(*)$ if some extra conditions are imposed on its elements. In this direction we prove the following:

Theorem 2.2.1. Let R be a ring with unity 1, such that there exists a positive integer $n \geq 1$ satisfying $x^n \in z(R)$. If commutators in R are $!n$-torsion free, then R is neccessarily commutative.

Actully in case $n = 2$ or 3, the result holds if the commutators in R are 2 - torsion free or 3 - torsion free respectively. The proof in these cases is simple. We just replace x by $(1+x)$ in the identity $[x^2, y] = 0$ or $[x^3, y] = 0$, for all $y \in R$, as the case may be and get the result. However, in general case we need the following result due to Tong [121].

Lemma 2.2.1. Let R be a ring with unity 1, Suppose for $x \in R$

$$I_o^r(x) = x^r$$

and

$$I_k^r(x) = I_k^r(1+x) - I_{k-1}^r(x), \text{ if } k \geq 1$$

Then

$$I^r_{r-1}(x) = \tfrac{1}{2}(r-1)!r+!rx$$

$$I^r_r(x) = !r$$

and

$$I^r_j(x) = 0 \text{ for } j > r$$

Proof of Theorem 2.2.1. Since x^n is central, we have $[x^n, y] = 0$ for all $x, y \in R$ which using the symbols of lemma 2.2.1, R can be written as follows :

(2.1) $$[I^n_o(x), y] = 0, \quad \text{for all } x, y \in R$$

On replacing x by $(1+x)$ in (2.1) and using the lemma 2.2.1 we get,

$$[I^n_o(x) + I^n_1(x), y] = 0, \quad \text{for all } x, y \in R$$

i.e.,

$$[I^n_o(x), y] + [I^n_1(x), y] = 0 \quad \text{for all } x, y \in R$$

This together with (2.1) gives

(2.2) $$[I^n_1(x), y] = 0, \quad \text{for all } x, y \in R.$$

Now replace x by $(1+x)$ in (2.2) and use the lemma 2.2.1 and repeat the process $(n-1)$ times, to obtain,

(2.3) $$[I^n_{n-1}(x), y] = 0, \quad \text{for all } x, y \in R$$

Again this, in view of lemma 2.2.1 (323) yields:

$$[\frac{1}{2}(n-1)!n+!nx, y] = 0$$

i.e.,

$$!n[x, y] = 0$$

As the commutator in R are $!n$ - torsion free, we get $[x, y] = 0$ and hence R is commutative. □

§2.3.

The condition $(*)$ given in the previous section can be further weakened by assuming that some powers of the products of each pair of ring elements are central. In this section we shall consider the condition $(*)$ rather in a more general setting.

$(**)$: For any pair of elements $x, y \in R$, there exists an integer $n = n(x, y) > 1$, depending on x and y both such that $[(xy)^n, x] = 0$.

The following example shows that even a ring with unity satisfying the condition $(**)$ may be badly non commutative.

Example 2.3.2. Let D_k be the ring of $k \times k$ matrices or $GF(2)$, the Glois field of order 2 and $A_k = \{(a_{ij}) \in D_k \mid a_{ij} \in GF(2)$ such that $a_{ij} = 0$ for $i \geq j\}$. Consider $R = aI + A_3$ where $a \in GF(2)$ and I is 3×3 identity matrix. Then R is a non commutative ring with unity satisfying $[(xy)^2, x] = 0$ for all $x, y \in R$.

Despite the existence of examples such as above, we could succeed in proving the following.

Theorem 2.3.2. Let R be a ring with unity 1 satisfying the condition.

$(***)$: For any pair of elements $x, y \in R$, there exists a positive integer $n = n(x,y) \geq 1$ such that $[(xy)^n, x] = 0$ and $[(xy)^{n+1}, x] = 0$.

Then R must be commutative.

We know that cancellation in rings is not permissible in general. However, in special types of rings like division rings and domains, it is allowed but these constitute a relatively small classes of rings. We can nevertheless, invoke some limited cancellation properties in rings with unity 1. One of such properties is essentially proved in [97] which in a rather general setting will be reproved here. Infact, we prove the following lemma when the indices n are localized depending on the pair of ring elements x and y. Also the proof presented here is a shorter and simpler one.

Lemma 2.3.2. Let R be a ring with unity 1 and $f : R \longrightarrow R$ be any polynomial function of two variables with the property that $f(x+1, y) = f(x,y)$ for all $x, \in y \in R$.

(i) If for every pair of elements x and y of R there exists an integer $n = n(x,y) \geq 1$ such that $x^n f(x,y) = 0$, then neccessarily $f(x,y) = 0$.

(ii) If for every pair of elements x and y of R there exists an integer $n = n(x,y) \geq 1$ such that $f(x,y)x^n = 0$, then neccessarily $f(x,y) = 0$.

Proof.(i) Corresponding to $1 + x$ and y there exists an integer $m = m(1+x, y) \geq 1$ with $(1+x)^m f(x, y) = 0$. Choose $N = max(n, m)$. Then $x^N f(x, y) = 0$ and $(1+x)^N f(x, y) = 0$. Now multiplying $1 = \{(1+x) - x\}^{2N+1}$ on the right by $f(x, y)$, we get

$$f(x, y) = \{(1+x) - x\}^{2N+1} f(x, y)$$

On expanding the right hand side expression by binomial theorem and using the fact that $x^M f(x, y) = (1+x)^M f(x, y) = 0$ for all $M \geq N$, we obtain that $f(x, y) = 0$. □

(ii) Argue on the same lines as above and get the result. □

Before proceeding further, we pause for a moment to make the following pertinent observations.

Remark 2.3.1. The commutator function $f : x \to [x, y]$ possesses the property,

$$f(x+1) = [x+1, y] = [x, y] = f(x, y)$$

which is indeed that of above lemma.

Thus we can rewrite Lemma 2.3.2. in a particular setting :

Lemma 2.3.3. Let R be a ring with unity 1. If for any pair of elements x, y in R there exists a positive integer $n = n(x, y)$ such that $x^n[x, y] = 0$ or a positive integer $m = m(x, y)$ such that $[x, y]x^m = 0$, then neccessarily $[x, y] = 0$.

We devlop the proof of our theorem 2.3.2 step by step by establishing the following intermediatory results.

Lemma 2.3.4. If R is a division ring satisfying the condition $(***)$, then R is commutative.

Proof Let $0 \neq x \in R$. Then in new of $(***)$, for any $y \in R$, there exists an integer $n = n(x, x^{-1}y) \geq 1$ such that

$$[(xx^{-1}y)^n, x] = 0$$

and

$$[(xx^{-1}y)^{n+1}, x] = 0$$

i.e.,

$$[y^n, x] = [y^{n+1}, x] = 0 \text{ for all } y \in R.$$

i.e.,

$$y^n x = xy^n \text{ and } y^{n+1}x = xy^{n+1}$$

Thus,

$$y^{n+1}x = xy^{n+1} = (xy^n)y = (y^n x)y$$

i.e.

$$y^n(yx - xy) = 0.$$

i.e.

$$y^n[y, x] = 0.$$

Now in view of lemma 2.3.3, we have $[y, x] = 0$ which forces that R is commutative. □

If $U(R)$ is the totality of all units, $J(R)$ is the Jacobson radical of R and $C(R)$ is the commutator ideal of R, then arguing on the same lines as in the proof of lemma 2.3.3, we can straightforward conclude.

Lemma 2.3.5. For any ring R with unity 1 satisfying $(***)$

$$J(R) \subseteq U(R) \subseteq C(R).$$

Notice that any homomorphic image of R satisfying $(***)$ satisfies $(***)$ and also that for any positive integer $k > 1$ *no complete matrix ring* D_k over a division ring D satisfies $(***)$. Indeed if we consider $x = e_{21}$ and $y = e_{12} + e_{21}$, then a simple check verifies our claim. Thus in view of the structure theory of primitive rings, we may assume that a primitive ring satisfying $(***)$ must be a division ring and hence commutative by lemma 2.3.4. So we have proved. □

Lemma 2.3.6. If R is a primitive ring satisfying the condition $(***)$, then R is commutative.

In the sequel, also we prove.

Lemma 2.3.7. Let R be a ring of theorem 2.3.2 and $J(R)$ be its Jacobson radical. Then $R/J(R)$ is commutative.

Proof. Since $R/J(R)$ is a subdirect sum of primitive rings which are commutative by lemma 2.3.6, $R/J(R)$ itself must be commutative.

Now we are well set to achieve our goal of this section.

Proof of theorem 2.3.2. If $n = 1$, then only one condition $[xy, x] = 0$ is sufficient to yield that R is commutative. Infact $[xy, x] = 0$ implies that $x[x, y] = 0$ and lemma 2.3.3 forces that R is commutative.

Suppose that $n > 1$. Applying lemma 2.3.5 and 2.3.7, the condition $[(xy)^n, x] = 0$ gives

$$(xy)^n - x^n y^n \in J(R) \in C(R)$$

In particular, we have

$$[(xy)^n - x^n y^n, x] = 0$$

i.e.,

$$[(xy)^n, x] = [x^n y^n, x], \text{ for all } x, y \in R.$$

Thus in view of the condition $[(xy)^n, x] = 0$, we find

$$[x^n y^n, x] = 0$$

This implies that $x^n[y^n, x] = 0$ and by lemma 2.3.3 we get $[y^n x] = 0$. Further considering the condition $[(xy)^{n+1}, 0] = 0$ and proceeding on the same lines we get $[y^{n+1}, x] = 0$. So

$$y^{n+1}x = xy^{n+1} = (xy^n)y = (y^n x)y$$

i.e.,

$$y^n[y, x] = 0$$

This in view of lemma 2.3.3 yields that $[y, x] = 0$ and thus R is commutative. □

Remark 2.3.2. Example 2.3.2 is sufficient to demonstrate that the above result does not hold if $[(xy)^n, x] = 0$ but $[(xy)^{n+1}, x] \neq 0$. In fact, the non commutative ring given in the mentioned example satisfies $[(xy)^2, x] = 0$ for all ring elements x an y but considering $x = I + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and $y = I + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ we find that $[(xy)^3, x] \neq 0$.

§2.4

In 1961, Herstein [55] proved the following interesting result.

Theorem H_3. Let R be a ring and $n > 1$ be a fixed positive integer. If the map $x \to x^n$ defines a group endomorphism for the additive group R^+, then the commutator ideal of R must be nil and the totality of nilpotent elements of R is an ideal.

Infact, the hypothesis of the theorem H_3 can be simply formulated as follows

(H) : There exists a fixed positive integer $n > 1$ such that $(x+y)^n = x^n + y^n$, for all $x, y \in R$

This gives,

$$\begin{aligned}(x+y)^{n+1} &= (x+y)^n(x+y) \\ &= (x^n+y^n)(x+y)\end{aligned}$$

i.e.,

$$(x+y)(x+y)^n = (x^n+y^n)(x+y)$$

i.e.,

$$(x+y)(x^n+y^n) = (x^n+y^n)(x+y)$$

i.e.,

$$xy^n + yx^n = x^n y + y^n x$$

i.e.,

(B) : $$[x, y^n] = [x^n, y], \text{ for all } x, y \in R.$$

Thus the class of rings satisfying B includes the class of rings satisfying (H). However, there exist sufficient rings satisfying (B) which fail to satisfy (H). Motivated by this observation, H. E. Bell [27, Theorem 2] and [30, Theorem 5] extended the theorem H_3 and established the following:

Theorem B_1 . Let n be a fixed positive integer and R be a ring generated by n-th powers of its elements. If R satisfies the identity (B), then R must be commutative.

Theorem B_2 . Let $n > 1$ be a fixed positive integer and R be a ring with unity 1 satisfying the identity (B). If the additive group R^+ of the ring is n-torsion free, then R must be commutative.

During the last two decades many algebraists investigated the commutativity of rings with similar identities. In this section we continue these investigations about the rings satisfying some weaker conditions when the exponents are allowed to vary with the elements of the rings. Infact, we consider the following condition.

(P) : let $n > 1$ and q be fixed non negative integers and R be a ring in which for each $y \in R$, there exist integers $m = m(y) \geq 0$ and $r = r(y) \geq 0$ such that $x^q[x^n, y] = [x, y^m]y^r$ for all $x \in R$.

Evidently, condition (P) is a wider generalization of the identity (B). It is natural to ask whether (B) can be replaced by (P) in the hypotheses of Theorem B_1 or Theorem B_2. Working in this direction, we prove the following:

Theorem 2.4.3. Let R be a ring with unity 1 satisfying the condition (P). If R is n-torsion free, then R must be commutative.

In our attempt to equip ourselves to prove the above result, we shall need the following lemmas essentially proved in [75], [68] and [32] respctivily.

Lemma 2.4.8. Let f be a polynomial in non commuting indeterminates $X_1, X_2, \ldots, X_n$ with relatively prime integral coefficients. Then, the following statements are equivalent.

(a) Every ring satisfying the polynomial identity $f = 0$ has nil commutator ideal.

(b) For every prime p, the ring $(GF(p))_2$ of 2×2 matrices over $GF(p)$, the Glois field of order p fails to satisfy $f = 0$.

(c) Every semi prime ring satisfying $f = 0$ is neccessarily commutative.

Lemma 2.4.9. If $x, y \in R$ and $[x, [x, y]] = 0$, then for all positive integers m,

$$[x^m, y] = mx^{m-1}[x, y].$$

Lemma 2.4.10. Let R be a ring. If for each $x, y \in R$, there exists a polynomial $f(\lambda) \in \lambda\mathbb{Z}(\lambda)$ over the ring $\mathbb{Z}$ of integers such that $[x,y] = [x,y]f(\lambda)$, then R is commutative.

Now we begin to prove

Lemma 2.4.11. Let R be a ring satisfying the condition (P). Then $C(R) \subseteq N(R)$.

Proof. If $m > 0$, then replacing x by $x + y$ we get

$$(x+y)^q[(x+y)^n, y] = [x+y, y^m]y^r = [x, y^m]y^r$$

This together with the condition (P) gives

$$x^q[x^n, y] = [x, y^m]y^r, \text{ for all } x, y \in R$$

This is a polynomial identity and choice of $x = -e_{11} + e_{12}$ and $y = e_{11}$ shows that for every prime p, the ring of 2×2 matrices over $GF(p)$ fails to satisfy the above identity. Hence in view of lemma 2.4.8, we have $C(R) \subseteq N(R)$.

If $m = 0$, R satisfies the polynomial identity $x^q[x^n, y] = 0$ and the same choice of matrices over $GF(p)$ as above will establish that $C(R)$ is a nil ideal.

Hence in every case $C(R) \subseteq N(R)$. $\square$

Lemma 2.4.12. Let R be an n-torsion free ring with unity 1 satisfying the condition (P). Then nilpotent elements of R are central.

Proof. Let $N(R)$ be the set of nilpotent elements of R and $a \in N(R)$.

Then there exists a least positive integer $j = j(a)$ such that

$$a^k \in z(R), \quad \text{for all integers } k \geq j \tag{2.4.1}$$

If $j = 1$ for each $a \in N(R)$, then result holds trivially. Therefore assume that $j > 1$. Putting $x = a^{j-1}$ in (P) we get for all $y \in R$,

$$a^{q(j-1)}[a^{n(j-1)}, y] = [a^{j-1}, y^m]y^r$$

Now using (1) and the fact that $n(j-1) \geq j$ for $n > 1$ we find

$$[a^{j-1}, y^m]y^r = 0 \tag{2.4.2}$$

Again putting $x = 1 + a^{j-1}$ in (P), we have

$$\begin{aligned} (1 + a^{j-1})^p[(1 + a^{j-1})^n, y] &= [1 + a^{j-1}, y^m]y^r \\ &= [a^{j-1}, y^m]y^r \\ &= 0, \quad \text{in view of (2)} \end{aligned}$$

But $1 + a^{j-1}$ is invertible and so,

$$[(1 + a^{j-1})^n, y] = 0$$

This gives,

$$[1 + \binom{n}{1}a^{j-1} + \binom{n}{2}(a^{j-1})^2 + \cdots + a^{n(j-1)}, y] = 0$$

This together with (1) forces that

$$[1 + na^{j-1}, y] = 0$$

i.e.,

$$n[a^{j-1}, y] = 0$$

Since the elements of R are n-torsion free, we have

$$[a^{j-1}, y] = 0, \quad \text{for all } y \in R$$

Thus $a^{j-1} \in z(R)$. This contradicts that j is the least for (1) which leads to the conclusion that $j = 1$ forcing $a \in Z(R)$.
Hence,

$$N(R) \subseteq z(R).$$

This proves our lemma. □

Now we turn to the proof of our main result.

Proof of theorem 2.4.3. By lemma 2.4.12 with lemma 3.4.11, we have

$$C(R) \subseteq N(R) \subseteq Z(R). \tag{1}$$

For $m = 0, (P)$ reduces to

$$x^q[x^n, y] = 0$$

Now in view of (1) we can apply lemma 2.4.9 to obtain

$$x^q n x^{n-1}[x, y] = 0$$

i.e.,

$$x^{q+n-1}[nx, y] = 0$$

In view of lemma 2.3.3, this at once gives,

$$[nx, y] = 0 = n[x, y]$$

Hence, torsion condition on R forces $[x, y] = 0$.
Assume that $m \geq 1$, Replace x by $1 + x$ in (P) to get,

$$\begin{aligned}(1+x)^q[(1+x)^n, y] &= [1+x, y^m]y^r \\ &= [x, y^m]y^r \\ &= x^q[x^n, y]\end{aligned}$$

Again in view of (1), we can apply lemma 2.4.9 for both side expressions and get

$$(1+x)^{q+n-1}n[1+x,y] \;=\; x^{q+n-1}[nx,y]$$

i.e.,

$$(1+x)^{q+n-1}n[x,y] \;=\; x^{q+n-1}n[x,y]$$

i.e.,

$$n[(1+x)^{q+n-1}-x^{q+n-1}][x,y] \;=\; 0.$$

Thus because of torsion condition on R, we have

$$[(1+x)^{q+n-1}-x^{q+n-1}][x,y] \;=\; 0$$

i.e.,

$$[1+xf(x)][x,y] \;=\; 0 \;\text{ for some } f(x)\in \mathbb{Z}[x]$$

i.e.,

$$\begin{aligned} [x,y] &= xf(x)[x,y] \\ &= [x,y]xf(x), \;\text{ by } (1) \\ &= [x,y]F(x) \text{ for some} F(x)\in x\mathbb{Z}[x]. \end{aligned}$$

Now application of lemma 2.4.10, gives that R is commutative. This completes the proof of our theorem 2.4.3. □

Remark 2.4.3. A close look to the proofs of lemma 2.4.12 and theorem 2.4.3 reveals that the hypothesis of the theorem can be further weakened by assuming only commutators of R to be n-torsion free.

Remark 2.4.4. If we take $p = r = 0$ and $m = m(g) = n$, then we can deduce theorem B_2 due to Bell [75] as a Corollary of our theorem 2.4.3.

Now we supply some examples to justify the restrictions on the hypotheses of theorem 2.4.3.

Example 2.4.3. The ring of 3×3 upper triangular matrices over the ring $\mathbb{Z}$ of integers satisfies the identity $[x^n, y] = [x, y^n]$ for all $x, y \in R$ and all $n \geq 2$. However R is not commutative. This justifies that existence of unity 1 in R is essential in our result.

Example 2.3.4. Let S be the subring generated by the elementry matrices e_{12}, e_{13} and e_{23} in the ring 3×3 matrices on $\mathbb{Z}_p$, the field of integers mod p and let

$$R = \{(x, \bar{m}) \mid x \in S', \bar{m} \in \mathbb{Z}_2\}$$

Define addition and multiplication in R by

$$(x_1, \bar{m}_1) + (x_2, \bar{m}_2) = (x_1 + x_2, \bar{m}_1 + \bar{m}_2)$$

$$(x_1, \bar{m}_1)(x_2, \bar{m}_2) = (x_1 x_2 + \bar{m}_1 x_2 + \bar{m}_2 x_1, \bar{m}_1 \bar{m}_2)$$

It is readily verified that R is a ring with unity satisfying the condition (P). Indeed $[x^n, y] = [x, y^n]$ for all $x, y \in R$ and $n \geq 2$. However R is non commutative. This demonstrates that the restriction on the elements of R (more specifically on the commutators of R) to be n-torsion free is not superfluous.

In Chapter 4, we shall give an example (cf. example 4.5.1) of non-commutative semi-prime nil ring due to Baer in which it is straight forward to cheek that the ring satisfies the condition

$$x^p[x^{n(x)}, y] = [x, y^{m(y)}]y^r$$

This demonstrates that in the hypotheses of our theorem 2.4.3, we can not assume both the indices m and n as localized, rather at least one of them must be taken as fixed even if the ring is assumed to be semi-prime.

CHAPTER - 3

COMMUTATIVITY OF CERTAIN s-UNITAL RINGS

§3.1. INTRODUCTION

We have seen in the previous chapter that certain conditions which render a ring with unity commutative fail to yield commutativity for the rings without unity (compare example 2.2.1 with theorem 2.2.1). Many more such examples can be found in the existing literature. Despite these bad examples one needs not give up the case when the ring does not contain unity. Recently many workers like Tominaga, Hirano, Kamatsu, Kobayashi and Mogami etc. initiated the study of s-unital and one sided s-unital rings which constitute comparatively larger classes of rings than the class of rings with unity.

Following Tominaga [117], a ring R is said to be left s-unital (resp. right s-unital) if $x \in Rx$ (respectively $x \in xR$) and R is called s-unital if $x \in Rx \cap xR$, for all $x \in R$. It is immediate to notice that a ring with unity 1 is neccessiarily both left s-unital and right s-unital and hence s-unital ring. However, there are enough one sided s-unital and s-unital rings which do not contain unity. Also on the strength of examples 3.3.1 and 3.3.2, we can claim that a left s-unital ring need not be a right s-unital and vice versa. In the present chapter we shall establish some commutativity theorems for these wider classes of rings under some appropriate conditions.

Section 3.2 has been devoted to a few preliminaries and basic results related to one sided s-unital and s-unital rings. In section 3.3, the result proved in theorem 2.4.1 has been extended to left s-unital rings whereas in section 3.4, we have obtained an extension of the same result for right s-unital rings. In section 3.5, some more commutativity theorems have been established for these classes of rings using Streb's classification of non-commuative rings [116]. Examples are also provided to show that some of our results proved for left s-unital rings cannot be established for right s-unital rings and vice versa.

§3.2.

In a paper [117], Tominaga proved the following result about s-unital (respectively left or right s-unital) rings.

Lemma 3.2.1. If R is an s-unital (resp. left or right s-unital) ring, then for any finite subset F of R, there exists an element e in R such that $ex = xe = x$ (resp. $ex = x$ or $xe = x$), for all $x \in F$.

Such an element e is called a Pseudo identity (resp. Pseudo left identity or Pseudo right identity) of F in R.

Following [64], a ring property P is called an h-property if P is inherited by every subring and every homomorphic image of the ring R. More weakly, if P is inherited by every finitely generated subring and every natural homomorphic image modulo the annihilator of a central element, then P is called an H - property. Finally, a ring property P such that the ring has the property P if and only if all its finitely generated subrings have the property P, is called an F-property.

The following result is essentially proved in [64, Proposition 1] which enables us to reduce some problems of s-unital rings into those of rings with unity 1.

Lemma 3.2.2. Let P be an H-property and P' be an F-property. If every ring with unity 1 having the property P has the property P' then every s-unital ring having P has P'.

§3.3

The example 2.4.1 in the previous chapter is sufficient to point out the existence of non - commutative rings without unity satisfying the hypotheses of the theorem 2.4.1. However, the same result may be extended to the classes of one sided s-unital rings imposing some restrictions on the indices.

We begin with the following.

Theorem 3.3.1. Let R be a left s-unital ring satisfying the condition.

(P^*) : For all x, y in R, there exist fixed non negative integers $n \geq 1, q \geq 0$, $m = m(y) \geq 0$ and $r = r(y) \geq 0$ with $(n, q, m, r) \neq (1, 0, 1, 0)$ such that $x^q[x^n, y] = [x, y^m]y^r$.

Moreover if commutators in R are n-torsion free, then R is s-unital.

Proof. Since R is left s-unital, for any $x, y \in R$, in view of lemma 3.2.1, we can choose an element $e \in R$ such that $ex = x$ and

$ey = y$. If $(m,r) \neq (1,0)$ then replacing x by e, we find that

$$e^q[e^n, y] = [e, y^m]y^r$$

i.e.,

$$e[e, y] = [e, y^m]y^r$$

i.e.,

$$ey - ye = ey^{m+r} - y^m ey^r$$

i.e.,

$$y = ye + y^{m+r} - y^m ey^r \in yR., \quad \text{for all } y \in R$$

On the other hand if $(m,r) = (1,0)$ then $(n,q) \neq (1,0)$ and condition (p^*) becomes $x^q[x^n, y] = [x, y]$. Now replace y by e to get

$$x^q[x^n, e] = [x, e]$$

$$x^q(x^n e - ex^n) = xe - ex = xe - x$$

i.e.,

$$x = xe - x^{q+n}e + x^q ex^n \in xR, \quad \text{for all } x \in R$$

Thus in every case R is right s-unital and hence s-unital. □

It is easy to observe that condition (P^*) is an h-property and condition "being commutative" is an F-property. Hence in view of lemma 3.2.2 together with theorem 3.3.1, we can assume that the left s-unital ring R satisfying (P^*) has unity 1. Thus theorem 2.4.3, yields the following.

Theorem 3.3.2 Let R be a left s-unital ring satisfying (P^*). If commutators of R are n-torsion free, then R is neccessarily commutative.

Proceeding on the same lines with respective variations, we can also prove the following.

Theorem 3.3.3 Let R be a right s-unital ring satisfying the condition:

(P^{**}) : There exist fixed non - negative integers $n \geq 1, q \geq 0$, and for each $y \in R$, there exists integers $m = m(y) \geq 0$, $r = r(y) \geq 0$, such that $[x^n, y]x^q = [x, y^m]y^r$ for all $y \in R$.

Moreover, if commutators in R are n-torsion free, then R is commutative.

One may find that our Theorems 3.3.2 and 3.3.3 together include many previously known results. To mention a few [5, Theorem 1], [61, Theorem] and [61, Corollary].

The following examples demonstrate that a left s-unital ring satisfying condition (P^{**}) or a right s-unital ring satisfying condition (P^{*}) need not be commutative even if the ring is n-torsion free.

Example 3.3.1 Let

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

be a subring of the ring of all 2×2 matrices over $GF(2)$. Then R is a non commutative left s-unital ring which satisfies (P^{**}) for all fixed positive integers $n > 1, m, p$ and r. Also the ring R is n-torsion free for any odd integer.

Example 3.3.2 Let

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$$

be a subring of the ring of all 2×2 matrices over $GF(2)$. Then R is a non commutative right s-unital ring satisfying the property (P^*) for any fixed positive integers $n > 1, m, q$ and r. Evidently R is n-torsion free for all odd integers.

§3.4

The study of Boolean condition ($x^2 = x$) has been extended in different directions (cf. [12], [94], [104] and [114]. In view of a result of Searcoid and Mac-Hale [114], a ring in which products of all pairs of ring elements are idempotents is neccessarily commutative. Obviously such a ring satisfies the conditions :

(i) $xy = xy^2x$, for all x, y in R

(ii) $yx = xy^2x$, for all $x, y \in R$

Commutativity of a ring satisfying either of the conditions (i) and (ii) is easy to establish. Although example 2.4.1 rules out the possibility that the conditions $xy - xy^2x \in Z(R)$ or $yx - xy^2x \in Z(R)$, for all $x, y \in R$ may also yield commutativity in arbitrary rings, recently Ashraf et.al [18] proved the following result in a more general setting for rings with unity 1.

Lemma 3.4.3. Let R be a ring with unity 1 and $n > 1$ be a positive integer satisfying either of the conditions

$$(*): \qquad [xy - xy^n x, x] = 0, \quad \text{for all } x, y \in R$$

$$(**): \qquad [yx - xy^n x, x] = 0, \quad \text{for all } x, y \in R$$

Then R must be commutative.

The existence of enough non commutative rings R without unity with $R^2 \subseteq Z(R)$ strengthens the existence of unity in the hypothesis of lemma 3.4.1. However we have succeeded in generalizing the above result for one sided s-unital rings.

Theorem 3.4.4. Let R be a left s-unital ring satisfying the condition $(*)$. Then R must be commutative.

Theorem 3.4.5. Let R be a right s-unital ring satisfying the condition $(**)$. Then R must be commutative.

Though we shall generalize above theorems further in the next section, but we have prefered to prove them independly in order to exhibit a useful different techniques.

In preparation for establishing the above theorems, we state the following lemma proved in [120].

Lemma 3.4.4. Let R be a left (resp. right) s-unital ring. If for each pair of elements x, y of R there exists a positive integer $k = k(x,y)$ and an element $e' = e'(x,y)$ of R such that $x^k e' = x^k$ and $y^k e' = y^k$ (resp. $e'x^k = x^k$ and $e'y^k = y^k$), then R is s-unital.

As a matter of fact, the lemma 3.4.4 is a powerful criterion for one sided s-unital rings to be s-unital. Yet another striking criterion will be given in the subsequent section [cf. lemma 3.5.7].

Proof of theorem 3.4.4. Since R is left s-unital, by lemma 3.2.1 we can find an element $e \in R$ corresponding to any $x, y \in R$ such that $ex = x$ and $ey = y$. Now replacing y by e in the condition (*), we get

$$[xe - xe^n x, x] = 0$$

This gives,

$$xex - xe^n x^2 - x^2 e + x^2 e^n x = 0$$

i.e.,

$$x^2 = x^2 e (\text{ noting that } xex = x^2)$$

$$\text{and } x^2 e^n x = x^2 e^{n-1}(ex) = x^2 e^{n-1} x, \text{ etc.}$$

Next, putting $x = e$ in (*), we get

$$[ey - ey^n e, e] = 0$$

This gives

$$y = ye \text{ i.e., } y^2 = y^2 e$$

Thus by lemma 3.4.4, R is s-unital and in view of lemma 3.2.2, we may assume that R has unity 1. Hence lemma 3.4.3, forces that R must be commutative. □

Also, applying the same techniques with neccessary variations, we can establish theorem 3.4.4. □

§3.5

In a paper [116], W. Streb provided a classification of all non-commutative rings by establishing the following :

Lemma 3.5.5. Let P be a ring property which is inherited by factor subrings . Then every ring R with unity 1 is commutative if no subrings either of the following types satisfy P

(a): $R = \begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(P) \end{pmatrix}$, where p is a prime

(b) : $M_\sigma(F) = \left\{ \begin{pmatrix} a & b \\ 0 & \sigma(b) \end{pmatrix}, \mid a, b \in F \right\}$ F is a finite field with a non trivial automorphism σ

(c): A non-commutative division ring.

(d): $T =< 1 > +S$ where S is a non-commuative radical subring of T

(e): $T =< 1 > +S$ where S is a non-commuative subring of T such that $S[S,S] = [S,S]S$

The following lemma is proved in [72].

Lemma 3.5.6. Let R be a non-commuative ring with the property that for each $x, y \in R$, there exist polynomials $f(\lambda), g(\lambda) \in \lambda^2\mathbb{Z}(\lambda)$ such that $[x - f(x), y - g(y)] = 0$. Then there exists a factor subring of R which is of the type (a) or (b).

In this section we shall extend our theorems 3.4.4 and 3.4.5 using the classifications of non-commutative rings given by Streb [cf. lemma 3.5.5].

Theorem 3.5.6. A ring with unity 1 satisfying any one of the following properties is necessarily commutative :

(P_1) : For all $x, y \in R$, there exists polynomial $f(\lambda) \in \lambda^2\mathbb{Z}(\lambda)$ such that $[yx^n - x^m f(y)x^q, x] = 0$ where n, m and q are fixed positive integers.

(P_2) : For all $x, y \in R$, there exists polynomial $f(\lambda) \in \lambda^2\mathbb{Z}(\lambda)$ such that $[x^n y - x^m f(y)x^q, x] = 0$ where n, m and q are fixed non negative integers.

Before proving the above theorem, we may pause to note that the conditions (P_1) and (P_2) of our theorem are wider generalizations of conditions $(*)$ and $(*)$ respectivily.

Proofof theorem 3.5.6. We shall prove our theorem for rings satisfying (P_1) and for rings satisfying (P_2) will follow on the same lines.

If n, m and $q = 0$, then condition (P_1) reduces to the condition of theorem 1.3.7 and consequently R turns out to be commutative. We shall therefore ignore this trivial case.

First consider the rings of the type (a) and let $x = e_{22},\ y = e_{12}$. Then

$$[e_{12}e_{22}^n - e_{22}^m f(y) e_{22}^q,\quad e_{22}] \neq 0.$$

Thus no rings of the type (a) satisfy (P_1).

Now, we consider the rings of type (b). Suppose x, y are elements of $M_\sigma(F)$ such that $x = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$ where $a_{22} = \sigma(a_{11}) \neq a_{11} \neq 0$ and $y = e_{12}$. Then

$$\begin{aligned}
[yx^n - x^m f(y) x^q, x] &= [yx^n, x] \\
&= [y, x]x^n \\
&= yx^{n+1} - xyx^n \\
&= (yx - xy)x^n
\end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} 0 & a_{22} - a_{11} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_{11}^n & 0 \\ 0 & a_{22}^n \end{pmatrix} \\
&= \begin{pmatrix} 0 & (a_{22} - a_{11})a_{22}^n \\ 0 & 0 \end{pmatrix} \\
&\neq 0,
\end{aligned}$$

for all nonnegative integers n, m, q and $f(\lambda) \in \lambda^2 \mathbb{Z}[\lambda]$. Hence R has no factor subring of type (b) satisfying (P_1).

Next, let R be a non-commutative division ring. If $0 \neq x \in R$, then for every $y \in R$, we have

$$\begin{aligned}
0 &= [yx^{-n} - x^m f(y)x^{-q}, x^{-1}], \text{ for some} f(y) \in y^2\mathbb{Z}(y) \\
&= x^{-1}(yx^{-n} - x^{-m}f(y)x^{-q}) - (yx^{-n} - x^{-m}f(y)x^{-q})x^{-1} \\
&= (yx^{-n} - x^{-m}f(y)x^{-q})x - x(yx^{-n} - x^{-m}f(y)x^{-q}) \\
&= x^m(yx^{-n} - x^{-m}f(y)x^{-q})x.x^{q+n} - x^m x(yx^{-n} - x^{-m}f(y)x^{-q})x^{q+n} \\
&= (x^m yx^q - f(y)x^n)x - x(x^m yx^q - f(y)x^n) \\
&= [x^m yx^q - f(y)x^n, x].
\end{aligned}$$

This gives that

$$x^m[yx^q, x] = [f(y), x]x^n. \tag{3.5.1}$$

Again for $f(y) \in R$, we have $t(f(y)) \in (f(y))^2\mathbb{Z}(f(y))$ such that

$$[f(y)x^n - x^m t(f(y))x^q, x] = 0.$$

This yields that

$$[f(y)x, x]x^n = x^m[t(f(y))x^q, x]. \tag{3.5.2}$$

Thus (3.5.1) and (3.5.2) together gives,

$$\begin{aligned}
x^m[yx^q, x] &= x^m[t(f(y))x^q, x] \\
&= x^m[h(y)x^q, x] \text{ where } h(\lambda) = t(f(y)) \in \lambda^2\mathbb{Z}(\lambda)
\end{aligned}$$

$$\text{i.e. } x^m[y, x]x^q = x^m[h(y), x]x^q \tag{3.5.3}$$

Since x is a nonzero element of a division ring R, (3.5.3) yields that

$$[y, x] = [h(y), x]$$

$$\text{i.e. } [y - h(y), x] = 0, \text{ for all } y \in R.$$

Hence in view of theorem 1.3.7, R is commutative. which is a contradiction. This shows that no rings of type (c) satisfy (P_1).

Further, let R have a factor subring $T =< 1 > +S$, where S is a non-commutative radical subring of T. Suppose $s_1, s_2 \in S$ so that $1 - s_1$ is a unit and hence as in the above, we can find $f(\lambda) \in \lambda^2 \mathbb{Z}(\lambda)$ such that $[s_2 - f(s_2), s_1] = 0$ and by theorem 1.3.7, S is commutative, a contradiction. Hence R is not of type (d).

Proceeding on the same lines with necessary variations, we can show that R satisfying (P_1) is also not of type (e).

Thus we have shown that no rings of types (a), (b), (c), (d) or (e) satisfy (P_1). Therefore by lemma 3.5.5 , R must be commutative. □

In [72], Kamatsu et al. proved the following conditions for a ring to be striclly one-sided s-unital.

Lemma 3.5.7. If R is a left s-unital ring and not right s-unital, then R must have a factor subring of the type

$$(a)_l : \begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}, \ p \text{ is a prime}$$

Similarly if R is right s-unital and not left s-uital, then R has a subring of the type

$$(a)_r : \begin{pmatrix} 0 & GF(p) \\ 0 & GF(P) \end{pmatrix}, \ p \text{ is a prime}$$

The above lemma helps us extend theorem 3.5.6 to one-sided s-unital rings. Indeed, if R is a left s-unital (resp. right s-unital) ring satisfying the property (P_1), then on the lines of the proof for theorem 3.5.6, it can be easily shown that no rings of type $(a)_l$ (resp. $(a)_r$) satisfy (P_1) and as a consequence of lemma 3.5.7, R is s-unital. Hence in view of lemma 3.2.2, a left s-unital (resp. right s-unital) ring satisfying (P_1) may be assumed to be a ring with unity 1 which by theorem 3.5.6 must be commutative.

Similarly we can prove the commmutativity of a one sided unital ring satisfyinf (P_2) and hence the following :

Theorem 3.5.7. If R is a one sided s-unital ring satisfying the property (P_1) or (P_2), then R must be commutative.

CHAPTER - 4

STRUCTURES OF CERTAIN PERIODIC RINGS AND COMMUTATIVITY

§4.1 INTRODUCTION

During the middle of 19^th century, English logician George Boole [39] introduced an algebra of sentences which was soon known as Boolean Algebra after his name. In an attempt to represent Boolean algebra with another equivalent system, M. H. Stone [115] introduced the concept of Boolean ring as a ring R in which every element is idempotent (i.e., $x^2 = x$ for all $x \in R$). Boolean rings are neccessarily commutative. Years ago, Jacobson [67] generalized the Boolean condition and proved a theorem which at the same time generalizes the classical theorem of Wedderburn [122] that every finite division ring is commutative. The theorem to which we refer is, namely: any ring in which $x^{n(x)} = x, n(x)$ an integer greater than 1, for every ring element x, must be commutative(cf. theorem 1.3.2). A ring satisfying Jacobson condition ($x^{n(x)} = x$) may be called J- ring. A concept that generalizes Boolean rings and J-rings is that of a periodic ring. in which there exist distinct positive integers n, m depending on the ring element x such that which $x^n = x^m$ for every element x. Periodic rings, in general are non commutative.

The present chapter concerns with determining the structure of certain periodic rings and obtaining their commutativity as a consequence.

In section 4.2, we collect some preliminary material about periodic rings. As mentioned in §3.4, Searcoid and MacHale [114] weakened the Boolean condition by establishing that a ring satisfying $xy = (xy)^{n(x,y)}$

for all pair of element x and y must be commutative. Later, Ligh and Luh [92] proved that such rings are the direct sum of J- rings and zero rings. In section 4.3, we obtain some decomposition theorems for rings under either of the conditions (i) $xy = f(y)x^n$ (ii) $xy = f(x)y^n$ (iii) $xy = x^n f(y)$ (iv) $xy = y^n f(x)$, where $f(t)$ is a polynomial in $t^2\mathbb{Z}[t]$ varying with the pair of elements x and y. Finally we deduce the commutativity of such rings.

In section 4.4 we continue this study and prove the decompositions of the rings when the underlying conditions are assumed to be satisfied by certain appropriate subset of the rings. The present chapter has been concluded by providing some examples in the last section to show that not all rings are periodic and also that not all periodic rings are commutative. The rings in these examples justify the conditions imposed on the theorems.

§4.2.

Perhaps, the classical theorem 1.3.2 due to Jacobson [67] as motivation the concepts of Potent elements and J-rings are introduced.

Definiton 4.2.1 (Potent Element). An element x of a ring R is called *Potent* if there exists a positive integer $n(x) > 1$ such that $x^{n(x)} = x$

Definition 4.2.2 (J-ring). A ring R is called *J-ring* if for every $x \in R$, there exists a positive integer $n = n(x) > 1$ such that $x^n = x$.

A concept that generalizes Boolean rings and J-rings is that of a periodic ring.

Definition 4.2.3 (Periodic Ring). A ring R is said to be *Periodic* if for each $x \in R$, there exist distinct positive integers $n = n(x)$ and $m = m(x)$ depending on x for which $x^n = x^m$.

One can easily notice that class of periodic rings includes Boolean rings also. Wheather a periodic ring is neccessarily Boolean is not known. However, it is well known that a ring with unity is Boolean if and only if $x^{n+1} = x^n$, for some positive integer $n \geq 1$. A sufficient condition for a ring to be periodic is the following Chacron's criterion:

Theorem ([43, Chacron]). Let R be a ring in which for each element x there exists an integer $m = m(x) > 1$ and a polynomial $f(t)$ over the ring $\mathbb{Z}$ of integers such that $x^m = x^{m+1}f(x)$. Then R is neccessarily periodic.

Howard E. Bell gave two remarkable structureal results for periodic rings.

Theorem B_3 ([29, Bell]). If R is any periodic ring. then R has each of the following properties

(a) Power of each element in R is idempotent.

(b) Each element $x \in R$ can be expressed in the form $x = a + b$ where a is potent and b is nilpotent.

Theorem B_4 ([33, Bell]). If in a periodic ring R, every element $x \in R$ has a unique representation as $x = a + b$ where $a \in P(R)$ and $b \in N(R)$ then $P(R)$ and $N(R)$ both are ideals and

$$R = P(R) \oplus N(R)$$

§4.3.

In 1986, Searcoid and MacHale [114] established commutativity of the rings satisfying the condition that all products of pairs of elements in a ring are potent. Further Ligh and Luh [92] using the mentioned result pointed out that such rings are direct sum of J-rings and zero rings. Motivated by these observations, we consider the following related ring properties and establish some results leading to direct sum decomposition theorems which also allow us to determine the commutativity of the rings.

Let $\mathbb{Z}(t)$ be the totality of all polynomials in t with coefficients from the ring $\mathbb{Z}$ of integers.

(P_1) : For each x, y in R, there exists an integer $n = n(x,y) \geq 1$ and $f(t) \in t^2\mathbb{Z}[t]$ such that $xy = f(y)x^n$.

$(P_1)^*$: For each x, y in R, there exists an integer $n = n(x,y) > 1$ and $f(t) \in t^2\mathbb{Z}[t]$ such that $xy = x^n f(y)$.

(P_2) : For each x, y in R, there exists an integer $n = n(x,y) > 1$ and $f(t) \in t^2\mathbb{Z}[t]$ such that $xy = y^n f(x)$.

$(P_2)^*$: For each x, y in R, there exists an integer $n = n(x,y) > 1$ and $f(t) \in t^2\mathbb{Z}[t]$ such that $xy = f(x)y^n$.

Notice that if y is replaced by x in either of the above conditions, $f(t) \in t^2\mathbb{Z}[t]$ guarantees that R satisfies the Chacron's criterion for periodicity and consequently the rings satisfying either of the properties (P_1) through $(P_2)^*$ are neccessarily periodic.

We begin with the following result.

Lemma 4.3.1. Let R be a ring satisfying either of the conditions (P_1) or (P_2). Then $RN(R) = N(R)R = (0)$.

Proof. First we suppose that R satisfies (P_1). Notice that R satisfying (P_1) is zero-commutative. Indeed if $xy = 0$, then there exist an integer $m' = m(y,x) \geq 1$ and $g(t) \in t^2\mathbb{Z}[t]$ such that $yx = g(x)y^{m'} = 0$. Now let $u \in N(R)$ and $x \in R$. Then there exist an integer $m_1 = m_1(x,u) \geq 1$ and $f_1(t) \in t^2\mathbb{Z}[t]$ such that $xu = f_1(u)x^{m_1}$. Choose $m_2 = m_2(f_1(u), x^{m_1}) \geq 1$ and $f_2(t) \in t^2\mathbb{Z}[t]$ such that $f_1(u)x^{m_1} = f_2(x^{m_1})(f_1(u))^{m_2}$. Similarly for the integer $m_3 = m_3(f_2(x^{m_1}), (f_1(u))^{m_2}) \geq 1$ and $f_3(t) \in t^2\mathbb{Z}[t]$ we have $f_2(x^{m_1})(f_1(u))^{m_2} = f_3((f_1(u))^{m_2})(f_2(u^{m_1}))^{m_3}$. Continuing this process we can easily see that

$$xu = f_t((f_{t-2}(\cdots(f_2(x^{m_1}))^{m_3})^{m_{t-1}})f_{t-1}(f_{t-2}(\cdots(f_1(u)^{m_2}\cdots)^{m_t})$$

or

$$xu = f_t((f_{t-2}(\cdots(f_1(u))^{m_2}\cdots)^{m_{t-1}})(f_{t-1}(f_{t-2}\cdots(f_2(u^{m_1}))^{m_3}\cdots)^{m_t}$$

according as t is even or odd. Since $m_t \geq 1$ and u is nilpotent, the above expressions, in every case yield that $xu = 0$ for all $x \in R$ and $u \in N$. Hence $N(R) = (0)$. Further, in view of zero-commutativity, we get

$$N(R)R = RN(R) = (0).$$

Similarly we can get the result in case R satisfies the condition (P_2). $\square$

Lemma 4.3.2. Let R be a ring of the above lemma and $P(R)$ be the totality of all potent elements of R. Then the following hold:

(i) for any $a \in P(R)$, some power of a is idempotent.

(ii) For any $a, b \in P(R)$, there exists an integer $k > 1$ such that

$$a^k = a \quad \text{and} \quad b^k = b.$$

Proof.(i) Since $a \in P(R)$, there exists an integer $p = p(a) > 1$ such that $a^p = a$. Now taking $e = a^{p-1}$ we have

$$\begin{aligned} e^2 &= (a^{p-1})^2 \\ &= a^{2p-2} \\ &= a^{p+(p-2)} \\ &= a^p a^{p-2} \\ &= a a^{p-2} \\ &= a^{p-2+1} \\ &= e \end{aligned}$$

(ii) Since $a, b \in P(R)$, there exist integers $p = p(a) > 1$ and $q = q(a) > 1$ such that $a^p = a$ and $b^q = b$. Now consider the integer,

$$\begin{aligned} k &= (p-1)q - (p-2) \\ &= (q-1)p - (q-2) \end{aligned}$$

Then

$$\begin{aligned} a^k &= a^{(p-1)q-(p-2)} \\ &= a^{(p-1)q-(p-1)+1} \\ &= a^{(p-1)(q-1)+1} \\ &= (a^{(p-1)})^{(q-1)} a \\ &= a^{p-1} a \text{ because, } a^{p-1} \text{ is idempotent by}(i) \text{ above} \\ &= a^p \\ &= a \end{aligned}$$

Similarly $b^k = b$.

we are now well equipped to prove the following structure of rings under the mentioned conditions.

Theorem 4.3.1. Let R be a ring satisfying either of the conditions (P_1) or (P_2). Then $R = P(R) \oplus N(R)$, where P is a J-ring and N is a nill ring.

Proof. In condition (P_1) replacing y by x, we notice that R satisfies Chacron's condition and hence R is periodic. Since R is periodic, every element $x \in R$ can be written in the form $x = a + u$ where $a \in P(R)$ and $u \in N(R)$ by theorem B_3. To complete the proof of the theorem now it is sufficient to show that the above representation is unique. Indeed, if $a + u = b + v$ for some $a, b \in P(R)$ and $u, v \in N(R)$, then

$$a - b = v - u, \text{ where } a, b \in P(R) \text{ and } u, v \in N(R) \tag{4.3.1}$$

Since $a, b \in P(R)$, in view of lemma 4.3.2 (ii) we can choose an integer $k > 1$ such that $a^k = a$ and $b^k = b$. Then by lemma 4.3.2 (i), $e_1 = a^{k-1}$ and $e_2 = b^{k-1}$ are idempotents with $e_1 a = a$ and $e_2 b = b$. Multiplying (4.3.1) by a and b separately on left, we have

$$\begin{aligned} a^2 - ab &= av - au \\ &= 0 \\ ba - b^2 &= (v - u)b \\ &= vb - ub \\ &= 0 \end{aligned}$$

Similarly multiplying (4.3.1) on right by a and b separately we get

$$a^2 - ba = 0 \text{ and } ab - b^2 = 0$$

Thus,

$$a^2 = ab = ba \text{ and } b^2 = ab = ba.$$

This yields that $a^2 = b^2$ for all $a, b \in P(R)$. Now

$$\begin{aligned} e_1 &= e_1^2 \\ &= (a^{k-1})^2 \\ &= (b^{k-1})^2 \\ &= e_2^2 \\ &= e_2 \end{aligned}$$

because a^{k-1}, $b^{k-1} \in P(R)$. Set $e_1 = e_2 = e$. Then multiplying (4.3.1) by e on the left we get $ea - eb = 0$ which forces that $a = b$. Thus we have shown that every element $x \in R$ has a unique representation as $x = a + b$, where $a \in P(R)$ and $b \in N(R)$. Hence in view of theorem B_4, $P(R)$ and $N(R)$ both are ideals and

$$R = P(R) \oplus N(R).$$

Similarly we can get the result in case R satisfies (P_2). □

Further, we notice that zero-commutativity is not implied by either of the conditions $(P_1)^*$ or $(P_2)^*$, but lemma 5.3.1 can be established even if R satisfies condition $(P_1)^*$ or $(P_2)^*$. We can therefore easily prove the following:

Theorem 4.3.2. Let R be a ring satisfying any one of the conditions $(P_1)^*$ or $(P_2)^*$. Then $R = P(R) \oplus N(R)$, where P is J-ring and N is a nil ring.

Also, in view of lemma 4.3.1 we conclude that the nilpotent elements of a ring satisfying either of the above conditions annihilate R on both sides and hence central. Since J-rings are commutative according to theorem 1.3.2, the consequence of the above theorems is the following corollary which generalizes many known results.

Corollary 4.3.1. Let R be a ring satisfying any one of the conditions (P_1) through $(P_2)^*$. Then R is neccesarily commutative.

§4.4

Working further on the above lines, one may ask the natural question as to what can one say about the direct sum decomposition of the ring R if the underlying conditions (P_1) or (P_2) are assumed to be satisfied by certain restricted elements of R. In this direction, we consider the following conditions for an apporiate subset S of R :

$(P_1)^{**}$ For each x, y in R/S, there exist an integer $n = n(x,y) > 1$ and $f(t) \in t^2\mathbb{Z}[t]$ such that $xy = x^n f(y)$.

$(P_2)^{**}$ For each x, y in R/S, there exist an integer $n = n(x,y) \geq 1$ and $f(t) \in t^2\mathbb{Z}[t]$ such that $xy = y^n f(y)$.

The following theorem is an attempt to answer the above question.

Theorem 4.4.3. Let R be a ring with $N(R) \neq \{0\}$ and S be an additive subgroup of R with $S \subseteq N(R)$. If for each $x, y \in R/S$ either of the conditions $(P_1)^{**}$ or $(P_2)^{**}$ holds, then $R = P(R) \oplus N(R)$, where P is a J-ring and N is a nil ring.

Proof. In condition $(P_1)^{**}$ replacing y by $x \in R/S$, we get

$$x^2 = x^n f(x) \text{ for all } x \in R/S \text{ and } n \geq 1.$$

(4.4.1) $$x^2 = x^m q(y), \text{ for all}, x \in R/S, \; q(y) \in \mathbb{Z}(y)$$

$$\text{and } m \geq 3 \text{ (taking } f(y) = x^2 q(y) \text{ and } m = n+2).$$

Since every element of S is nilpotent, we have for each $x \in S, x^{n'} = 0$ for some integer $n' > 1$ and we can write $x^{n'} = x^{n'+n''} = 0$ for any integer $n'' \geq 1$. Therefore, R is periodic by Chacron's criterion. Let $x \in N(R)/S$. Then application of (4.4.1) yields that $x^2 = 0$. Now our aim is to show that if $x \in N(R)/S, y \in R$ and $xy = 0$ then $yx = 0$. Suppose $y \in R/S$. Then it can be easily obtained by using condition $(P_1)^{**}$. Now assume $xz = 0$ for all $x \in N(R)/S$ and $z \in S$. Then $(x+z)x = 0 = x^2 + zx = zx$.

Hence we can claim our mentioned assumption. Next we shall prove that N is an ideal. Let $x, y \in S$. Then $x - y \in S \subseteq N$. Let $x \in N/S$. Then we have $x^2 r = 0$ for every $r \in R$ i.e. $x(xr) = 0$ whic gives $(xr)x = 0$ i.e.

(4.4.2) $$xRx = \{0\}.$$

Suppose $y \in N(R)$. Then $y^s = 0$ for some integer $s > 1$ and we have $(x-y)^{2s} = 0$. So in this case also we get $x - y \in N(R)$. Let $x \in N(R)/S$. Then using (4.4.2), we get $(xr)^2 = 0$ for every $r \in R$. Assume $z \in S$ and $x \in N(R)/S$. Then writting $zr = (x+z)r - xr$, yields that $zr \in N(R)$. Hence N is an ideal.

Let $x \in N/S$ and $y \in R/S$. Then by condition $(P_1)^{**}, xy = 0$. Using the fact that every element of S is a difference of two elements of R/S, we have $RN(R) = N(R)R = \{0\}$. Thus On the lines of the proof given in case of theorem 4.3.1, we get the required result i.e. $R = P(R) \oplus N(R)$.

As in the end of previous section 4.3, the above theorem yields the following corollary which generalizes the main theorem of Bell [23].

Corollary 4.4.2. Let R be a ring with $N(R) \neq \{0\}$ and A be an additive subgroup of R with $A \subseteq N$. If for each $x, y \in R/A$ either of the conditions $(P_1)^{**}$ or $(P_2)^{**}$ holds, then R is commutative.

§4.5.

There are two extreme cases for periodic rings: Boolean rings or in a general sceting J-rings and nil rings. Boolean rings as well as J-rings are neccessaily commutative (cf. theorem 1.3.2) while a nil ring even if it is semi-prime need not be commutative as is evident by the following example due to Baer [21].

Example 4.5.1. Let $\{G_i\}$ be a countable infinite family of cyclic groups with gensetors denoted by $g(0), g(1), \ldots, g(i), \ldots, g(-1), \ldots$ and $G = \oplus G_i$. Obviously G is an abelian group. Consider a mapping $F(i) : G \to G$ defined as follows :

$$g(j)F(i) = \begin{cases} 0 & , \quad \text{if } j = 0 \mod j^j \\ g(j-1) & , \quad \text{otherwise} \end{cases}$$

Then for each $i = 1, 2, \ldots, F(i)$ is an endomorphism of G.

Denote by R, the ring of endomorphisms of G which is generated by the endomorphisms $F(1), F(2), \ldots$. Now choose the non zero gensetors $0 \neq g(1) \neq g(2) \neq g(3)$. Then

$$\begin{aligned} g(3)(F(1)F(2)) &= (g(3)F(1))F(2) \\ &= g(2)F(2) \\ &= g(1) \\ &\neq 0 \end{aligned}$$

and

$$\begin{aligned} g(3)(F(2)F(1)) &= (g(3)F(2))F(1) \\ &= g(2)F(1) \\ &= 0 \end{aligned}$$

Thus $F(1)F(2) \neq F(2)F(1)$, for $F(1), F(2) \in R$ and hence R is non-commutative. Also it has been demonstrated in [21] that R is nil semi-prime ring.

Among the finite periodic rings is the following particular Corbas $(2, 2, \phi)$-ring [44] which may also be non-commutative.

Example 4.5.2. Let R be the additive direct sum $GF(2^2) \oplus GF(2^2)$ and ϕ be an automorphism of $GF(2^2)$. In R, define a multiplication as follows: For any (a, b) and $(c, d) \in R$;

$$(a, b)(c, d) = (ac, ad + b\phi(c))$$

It is easy to check that R is a periodic ring which is commutative only when ϕ is the identity automorphism

The following example demonstrates that in either of the conditions (P_1) through $(P_2)^*$ or $(P_1)^{**}$ and $(P_2)^{**}$ we can not assume $f(t) \in \mathbb{Z}(t)$.

Example 4.5.3. Consider R as the additive klein 4-group with elements, say $0, a, b$ and c. Define multiplication in R as follows :

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

Then R turns out to be a periodic ring satisfying the identity $xy = x^2y$, for all $x, y \in R$. However, R is not commutative.

CHAPTER - 5

CO-COMMUTATIVITY AND STRONG CO-COMMUTATIVITY PRESERVING MAPPINGS

§5.1. INTRODUCTION

A mapping from a ring R to itself is called centralizing on a non-void subset S of R if $[f(x), x]$ is central for all $x \in S$. In particular if $[f(x), x] = 0$ for all $x \in S$, the mapping f is described as commuting on S. It seems that E.C. Posner was the first to initiate during 1950's, the study of such mappings, precisely that of derivations (cf. definition 5.2.1). Over the last 25 years, several research workers have studied commutativity of prime and semi-prime rings admitting commuting and centralizing mappings, especially derivations and endomorphisms. There is also a growing literature on commutativity preserving (*cp*) and strong commutativity preserving (*scp*)-mappings(cf. definition 5.2.2 and definition 5.2.3 respectively). Inspired by these works we initiate the study of more general concepts than (*cp*) and (*scp*)-mappings. Infact, we introduce the concepts of co-commutativity preserving and strong co-commutativity preserving mappings. *Two mappings* $f, g : R \longrightarrow R$ *are said to be strong co-commutativity preserving* (*sccp*) *on a subset* S *of* R *if* $[x, y] = [f(x), g(y)]$, *for all* $x, y \in S$. The aim of the present chapter is to examine commutativity or near commutativity of prime and semi-prime rings when the mappings are derivations or endomorphisims.

In section 5.2, we shall recall some definitions and collect certain results related to our study. In section 5.3, we prove that if a semi-prime ring R with a nonzereo ideal A of R admits two (*sccp*)-derivations, then R contains a nonzero central ideal. In particular if R is assumed to be prime with the same property, then R must be commutative.

In section 5.4, we prove a theorem on $(sccp)$-endomorphisms. At places suitable examples are provided to justify some of the restrictions on the hypotheses of the results.

§5.2.

In this section we shall recall some important notions and collect certain related results already existing in the literature. Some of the material presented here may not be used in the subsequent sections but this has been included for the sake of completeness.

Definition 5.2.1 (Derivation). An additive mapping $d : R \longrightarrow R$ from a ring R to itself is called a *derivation* if d satisfies the property that $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$.

Definition 5.2.2 (Centralizing and Commuting Mappings). A mapping $f : R \longrightarrow R$ is called *centralizing* on a nonvoid subset S of R if $[f(x), x] \in Z(R)$, the center of R, for all $x \in S$.

In particular when $[f(x), x] = 0$ for all $x \in S$, the mapping f is said to be *commuting* on S.

Definition 5.2.3 (Commutativity Preserving Mapping). A mapping $f : R \longrightarrow R$ is called *commutativity preserving* (cp) on a nonvoid subset S of R if for any $x, y \in S$ whenever $[x, y] = 0$ then $[f(x), f(y)] = 0$ also.

Definition 5.2.4 (Strong Commutativity Preserving Mapping). Let R be a ring and $S \subseteq R$ be a nonvoid subset of R. A map $f : R \longrightarrow R$ is described as *strong commutativity preserving* (*scp*) on S if $[x, y] = [f(x), f(y)]$, for all $x, y \in S$.

Definition 5.2.5 (Essential Right Ideal). A right ideal K of the ring R is called *essential* if $K \cap A \neq (0)$ for every nonzero right ideals A.

Definition 5.2.6 (Lie and Jordan Structures). Given an associative ring R, we can induce on R using its operations two structures as follows:

(*i*) For all $x, y \in R$, the *Lie product* $[x, y] = xy - yx$.

(*ii*) For all $x, y \in R$, the *Jordan product* $x \circ y = xy + yx$.

Remark 5.2.1. For any $x, y, z \in R$, the following identities hold:

(*i*) $[xy, z] = x[y, z] + [x, z]y$.

(*ii*) $[x, yz] = y[x, z] + [x, y]z$.

(*iii*) $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$. This identity is generally known as *Jacobi identity.*

Theorem 5.2.1([99, Posner]). Let R be a prime ring of characteristic not 2 and d_1, d_2 be derivations of R such that iterate $d_1 d_2$ is also a derivation, then one at least of d_1, d_2 is zero.

Theorem 5.2.2([99, Posner]). Let R be a prime ring. If d is a nonzero derivation commuting on R, then R is commutative.

Theorem 5.2.3([99, Posner]). Let R be a prime ring. If R admits a nonzero derivation d which is centralizing on R, then R is commutative.

Theorem 5.2.4([36, Bell and Daif]). Let R be a semiprime ring and $A \neq 0$, a right ideal of R. If R admits a derivation d which is (scp) on A, then $A \subseteq Z(R)$, the center of R.

Theorem 5.2.5([36, Bell and Daif]). Let R be a prime ring and A be an essential right ideal of R. If R admits a non-identity endomorphism T which is *scp* on A, then R is commutative.

§5.3.

In 1994, Bell and Daif [36] investigated commutativity for prime and semi-prime rings admitting derivations or endomorphisms which are strong commutativity preserving (*scp*) on an appropriate subsets of the ring. Inspired by this study, we introduce the notions of co-commutativity preserving (*ccp*) and strong co-commutativity preserving (*sccp*) mappings and carry on the similar investigations in this direction.

Definition 5.3.7 (Co-Commutativity Preserving Mappings). Two mappings $f, g : R \longrightarrow R$ from R to itself are said to be *co-commutativity preserving* (*ccp*) on a nonvoid subset S of R if for any $x, y \in S$ whenever $[x, y] = 0$, then $[f(x), g(y)] = 0$ also.

Definition 5.3.8 (Strong Co-Commutativity Preserving Mappings). Two mappings $f, g : R \longrightarrow R$ from R to itself are said to be *strong co-commutativity preserving* (scp) on a nonvoid subset S of R if $[x, y] = [f(x), g(y)]$, for all $x, y \in S$.

Example 5.3.1. Let R be any ring with center $Z(R)$. Define a function $f : R \longrightarrow R$ given by $f(x) = cx + d(x)$, where $c \in Z(R)$ and d is a function from R to $Z(R)$. If I is the identity function on R, then f and I are (ccp)-maps on R.

For $(sccp)$ maps we refer to example 5.3.2.

The following lemmas are essentially proved in [37] and [61] respectively.

Lemma 5.3.1. Let A be a nonzero left ideal of a semiprime ring R. If d is a derivation on R which is nonzero on A such that $[x, d(x)] \in Z(R)$, for all $x \in A$, then R contains a nonzero central ideal.

Lemma 5.3.2. Let R be a semi-prime ring and $A \neq 0$ be a right ideal of R. If A is commutative as a ring, then $A \subseteq Z(R)$. If in addition R is prime, then R must be commutative.

Now, we establish the following result:

Theorem 5.3.6. Let R be a semi-prime ring and A a nonzero ideal of R. If R admits two strong co-commutativity preserving $(sccp)$-derivations, then R contains a nonzero central ideal.

Proof. Let d_1 and d_2 be two derivations which are $(sccp)$ on A. If either of these derivations is vanishing on A, then their $(sccp)$ property yields that $[x,y]=0$, for all $x,y \in A$ and hence in view of Lemma 5.3.2, we get the $A \subseteq Z(R)$. Thus we omit this trivial case and here onward assume that none of d_1 and d_2 is zero derivations on A. For all $x,y,z \in A$, we have $[xy,z] = [d_1(xy), d_2(z)]$. This gives that $x[y,z]+[x,z]y = [xd_1(y)+d_1(x)y, d_2(z)]$ which yields that

$$[x, d_2(z)]d_1(y) + d_1(x)[y, d_2(z)] = 0. \tag{5.3.1}$$

For any $r \in R$, replacing x by rx in (5.3.1) and using the same relation (5.3.1), we get $[r, d_2(z)]xd_1(y) + d_1(r)x[y, d_2(z)] = 0$. Now, put $r = d_1(z)$ in this expression to get $[d_1(z), d_2(z)]xd_1(y) + d_1^2(z)x[y, d_2(z)] = 0$. This implies that

$$d_1^2(z)x[y, d_2(z)] = 0. \tag{5.3.2}$$

Since R is a semi-prime ring it contains a family $P = \{P_i \mid P_i$ is a prime ideal of R with $\cap P_i = (0)\}$. For a typical $P_i \in P$, by (5.3.2) we find that $d_1^2(z)RI[y, d_2(z)] = 0 \subseteq P_i$. This gives that either $d_1^2(z) \in P_i$ or $A[y, d_2(z)] \subseteq P_i$.

Suppose that $d_1^2(z) \in P_i$. For a given $z \in A$, then from $[d_1(z)x, z] = [d_1(d_1(z)x), d_2(z)]$, we get $d_1(z)[x,y] + [d_1(z), z]x = [d_1^2(z)x, d_2(z)] + [d_1(z)d_1(x), d_2(z)]$. Thus $[d_1(z), z]x = [d_1^2(z)x, d_2(z)] \in P_i$ and we get

$$[d_1(z), z]A \subseteq P_i$$

On the other hand if $A[y, d_2(z)] \subseteq P_i$ i.e. $AR[y, d_2(z)] \subseteq P_i$ then $A \subseteq P_i$ or $[y, d_2(z)] \in P_i$ for all $y \in A$. But if $A \subseteq P_i$, then obviously $[d_1(z), z]A \subseteq P_i$. If $[y, d_2(z)] \subseteq P_i$ for all $y \in A$, then replacing y by ry for any $r \in R$, we get $[r, d_2(z)]y \in P_i$ i.e. $[R, d_2(z)]RA \subseteq P_i$ and consequently either $[R, d_2(z)] \subseteq P_i$ or $A \subseteq P_i$. But again if $A \subseteq P_i$, then obviously $[d_1(z), z]I \subseteq P_i$. Also, if $[R, d_2(z)] \subseteq P_i$, then relation $[d_1(ry), d_2(z)] = [ry, z]$ implies that $[ry, z] \in P_i$ i.e. $r[y, z] + [r, z]y \in P_i$. This together with $r[y, z] = r[d_1(y), d_1(z)] \in P_i$, yields that $[r, z]y \in P_i$ and so

$$[d_1(z), z] \subseteq P_i.$$

Therefore in both the cases we have $[d_1(z), z]A \subseteq P_i$ and so $[d_1(z), z]A \in \cap P_i = (0)$ and hence $[d_1(z), z]A[d_1(z), z] = 0$. Now, by semiprimeness of A we find that $[d_1(z), z] = 0$, and hence by Lemma 5.3.1, we get the required result. □

The following example demonstrates that under the hypotheses of Theorem 5.3.6 the commutativity of R can not be guarented.

Example 5.3.2. Let $R = R_1 \oplus R_2$, where R_1 is a non-commutative prime ring with derivation d_1 and R_2 is a commutative domain. Define $d : R \longrightarrow R$ by $d(r_1, r_2) = (d_1(r_1), 0)$. and $A = \{(0, r_2) \mid r_2 \in R_2\}$. so that A is an ideal in R. Then R is semiprime ring and d is a derivation which is $(sccp)$ on the ideal A consisting of elements of the form $(0, r_2)$.

However,in case of prime ring we prove the following:

Theorem 5.3.7. Let R be a prime ring which admits two $(sccp)$-derivations on R, then R is commutative.

Proof. Using the similar arguments as used in the begining of the proof of theorem 5.3.6, we find that $d_1^2(z)x[y, d_2(z)] = 0$, for all $x, y, z \in R$. Now, replacing y by $d_1(y)$, we get $d_1^2(z)x[d_1(y), d_2(z)] = 0$, consequently $d_1^2(z)R[d_1(y), d_2(z)] = d_1^2(z)R[y, z] = 0$. If R is a prime ring then for any prime ideal P_i of R, we have either $d_1^2(z) \in P_i$ or $[y, z] \in P_i$. But, since $[y, z] = [d_1(y), d_2(z)] = [d_1^2(y), d_2^2(z)]$ and using $d_1^2(z) \in P_i$ for all $z \in R$, we get $[y, z] \in P_i$. Hence in both the cases we get $[y, z] = 0$, for all $y, z \in R$. This completes the proof of theorem.□

§5.4.

We further extend the study for the rings admittings ($sccp$)-endomorphisms. In 1987, Bell and Martindale [37] proved that a semi-prime ring R with a nonzero left ideal A of R admitting a nontrivial endomorphism T which is one to one on A and centralizing on A, must be commutative. Later in 1994 Bell and Daif [36] generalized the mentioned result for scp-endomorphisms. In this section we continue the study and prove the following result. Hence onward, for the image $T(x)$ of $x \in R$ under an endomorphism T, we shall use the notation x^T.

Theorem 5.4.8. If R is a semi-prime ring and A a nonzero ideal of R. If R admits ($sccp$)-endomorphisms T_1 and T_2 then both T_1 and T_2 are commuting on A. Moreover, if T_1 is non-identical on $A \cap T_1^{-1}(A)$, then R contains a nonzero central ideal.

In preparation for proving our theorem, we need the following lemma due to Daif and Bell [45].

Lemma 5.4.3. Let R be a semi-prime ring and A be a nonzero ideal of R. If $z \in R$ centralizes $[A, A]$, then z centralizes A.

Proof of theorem 5.4.8. We have

$$[x^{T_1}, y^{T_2}] = [x, y], \text{ for all } x, y \in A. \tag{5.4.1}$$

Replacing x by x^2 in (5.4.1), we get

$$(x^{T_1} - x)[x, y] + [x, y](x^{T_1} - x) = 0. \tag{5.4.2}$$

Now, replace y by zy in (5.4.2), to get

$$(x^{T_1} - x)[x, z]y + (x^{T_1} - x)z[x, y] + [x, z]y(x^{T_1} - x) + z[x, y](x^{T_1} - x) = 0.$$

Since from equation (5.4.2), $(x^{T_1} - x)[x, y] = -[x, y](x^{T_1} - x)$, then the last equation reduces to

$$[x, z][y, x^{T_1} - x] - [z, x^{T_1} - x][x, y] = 0, \text{ for all } x, y, z \in A. \tag{5.4.3}$$

For any $r \in R$, substituting rz for z in (5.4.3) and using (5.4.3), we find that $[x, r]z[y, x^{T_1} - x] - [r, x^{T_1} - x]z[x, y] = 0$. Putting $y = x$ and $r = x^{T_1}$, this yields that $[x, x^{T_1}]z[x, x^{T_1}] = 0$, i.e. $[x, x^{T_1}]A[x, x^{T_1}] = 0$, for all $x \in A$. Thus semi-primeness of A forces that T_1 is commuting on A. Similarly, we can show that T_2 is also commuting on A.

Moreover, if T_1 is non-identical on $A \cap T_1^{-1}(A)$, then $[(xy)^{T_1}, y^{T_2}] = [xy, y]$ implies that $[x, y](y^{T_1} - y) = 0$. Replacing x by xz this gives that

$$[x, y]A(y^{T_1} - y) = 0, \text{ for all } x, y \in A. \tag{5.4.4}$$

Since R is a semi-prime ring it contains a family $P = \{P_i \mid P_i \text{ is a prime ideal of } R \text{ with } \cap P_i = (0)\}$. For a fixed $P_i \in P$, by (5.4.4), we obtain $[x, y]AR(y^{T_1} - y) \subseteq P_i$ and hence we get either $(y^{T_1} - y) \in P_i$, for all $y \in A$ or $[x, y]A \subseteq P_i$, for all $x, y \in A$. In the later case using the fact that A is a left ideal we get $[x,y]RA \subseteq P_i$ and we find that $[x,y] \in P_i$ for all $x, y \in A$. Now $\cap P_i = (0)$ together with the fact that P_i are ideals, we get

$$(y^{T_1} - y)[x,z] = 0 = [x,z](y^{T_1} - y), \text{ for all } x, y, z \in A.$$

That is $y^{T_1} - y$ centralizes $[A, A]$, so by Lemma 5.4.3, $y^{T_1} - y$ centralizes A. Now, set $V = A \cap T_1^{-1}(A)$. Then, for any $y \in V$ we have $y^{T_1} - y \in Z(A)$ and it forces by Lemma 5.3.2, that $y^{T_1} - y \in Z(R)$, for all $y \in V$. The hypothesis T_1 being non-identical on V, choose $y_o \in V$ such that $y_o^{T_1} - y_o \neq 0$, and let $U = A(y_o^{T_1} - y_o)$. Thus U is an ideal of R, and $y_o^{T_1} - y_o \in Z(R)$ implies that $0 \neq (y_o^{T_1} - y_o) \in U$.

Now, the equation

$$[x(y_o^{T_1} - y_o), y(y_o^{T_1} - y_o)] = [x,y](y_o^{T_1} - y_o) = 0, forall \text{ for all } x, y \in A.$$

This shows that $U \subseteq Z(R)$. Hence R contains a nonzero central ideal. □

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